Algebraic Thinking Within A Technology-Rich Learning Environment

Stephen Arnold
The University of Newcastle

Research into individual thinking about algebra faces a significant problem in the tacit nature of knowledge in this domain. This paper documents a research design which incorporated Repertory Grid principles within an image-based research model. Finely detailed study of perceptions of algebraic images offered a powerful complement to the more usual verbal approaches, providing insights into both student thinking about key concepts in algebra, and into the network of relations within which such concepts exist for individuals learning algebra.

Interviewer: .	..This person, towards the end of the lesson, nudges you again ... and says, "What's algebra?" We've had this one before, but how would you explain it to someone who didn't know?
J:	Letters?
Interviewer:	Then he says, "Letters?" Yeah, that's what we do in English.
J:	Um ... numbers and letters?
Interviewer:	Okay...
J:	Um ... a group of letters that mean something, equal something?
Interviewer:	Okay, a group of letters that mean something, equal something ... equal what? Equal numbers?
J:	Yeah.
Interviewer:	Okay, so a letter like "a" can stand for a number?
J:	Yeah.
Interviewer:	Okay, can it stand for more than one number?
J:	Yes.
Interviewer:	So it can stand for what? Two numbers?
J:	It can stand for ... I don't know ... anything.
Interviewer:	Anything?
J:	Any numbers.

The informant in this interview, J, is a female Year 10 student studying the Advanced course in mathematics for the New South Wales School Certificate. The transcript is revealing of the tacit nature of knowledge about algebra. It is clearly disjointed and difficult to articulate for the student and yet, upon probing, is essentially sound in its foundations. The important concept of variable as representing a range of values rather than simply a "placeholder" is present here, suggested at the end of the transcript. It was as a consequence of this recognition of the tacit nature of students' knowledge about algebra that alternative ways of gathering research data were explored, leading to the image-based methods described in this paper.

Student facility with key algebraic concepts such as "function" and "variable," in addition to the more traditional focus upon "equations," is increasingly recognised as central to an understanding of algebra across the years of secondary schooling and beyond, particularly within a technology-rich context (Harel & Dubinsky, 1992; Grouws, 1992; Romberg, Fennema & Carpenter, 1993). Such concepts are mathematically rich, capable of being thought of using not only verbal descriptors but a variety of alternative representations. Studies by Vinner and his colleagues over the past decade (Vinner, 1983, Tall & Vinner, 1989, Vinner & Dreyfuss, 1989) distinguish between a "concept image" ("the set of all the mental pictures associated...with the concept name, together with all the properties characterising them" [Vinner & Dreyfuss, 1989, p. 356]) and a "concept definition" ("a verbal definition that accurately explains the concept in a non-circular way" [Vinner, 1983, p. 293]). Such studies reveal, among other things, that concept images may not always be consistent with the formal definition, but that such inconsistencies are often not apparent. Much of the focus in this area has been upon identifying individual images which students prefer to use when thinking about functions, although the verbal descriptions given as definitions of functions frequently comprise multiple images. This was further supported in Arnold (1992), in which the pattern of representation was quite different when students were asked to describe a function "in their own words." In this situation, functions were most likely to be described as a "rule or relationship", which coincides with the common (non-mathematical) idea of "function", or one of several "multiple-image" definitions, such as an "algebraic object which can be graphed", or a "rule which can be expressed algebraically or graphically" (Arnold, 1992; p. 77). The relative richness of the imagery observed in such studies stands in marked contrast to the impoverished verbal descriptions which are commonly given to both teachers and researchers pursuing student understanding of algebraic concepts. A need was perceived to access this apparently significant source of information regarding the thinking of learners about their mathematics.

Method

The author has recently completed doctoral studies on the use of mathematical software tools by individuals for the learning of algebra. This study of learning to use new tools began as a case study of the teacher/researcher's interactions with a single student (referred to here as S) within a tool-rich algebraic learning context. The encounters occurred within individual tutorial situations over a period of almost two years, with some thirty-six hours of interactions recorded and analysed. As the study progressed, it grew to include five other student informants and two groups of preservice teachers as the cyclic nature of the grounded theory method demanded greater variability within the data, and new research questions and priorities became apparent. This paper describes the responses of three of the participants to a set of algebraic imagery tasks: **S**, a Year 12 student, attempting the high level Three Unit course in mathematics for the New South Wales Higher School Certificate; **T**, a Year 9 student, and **P**, a Year 7 student (who had not commenced his study of algebra at the time of this study).

In order to provide rich context for the study, detailed data were gathered concerning the thinking of participants regarding their perceptions of the nature of algebra and the ways in which it is best learned. As a result of the observations above it was decided to supplement verbal data with student responses to algebraic images, using a modified Repertory Grid (or *RepGrid*) technique, based upon Kelly's (1955) Personal Construct theory. The Repertory Grid was developed as a technique for eliciting, not only components of individuals' thinking about complex concepts, but aspects of the relationships between these components. It has been especially popular as a tool for investigating teacher and student thinking in educational research, offering an attractive blend of data which is both detailed and idiosyncratic in its reflection of individual responses, while at the same time potentially generalisable and amenable to statistical analysis (Solas, 1993; p. 209).

A common format for *RepGrid* analysis involves deriving a series of statements or prompts related to the particular construct in question (e.g. "good teaching"), then presenting these three at a time (randomly selected) and having the participants describe "in what way two of these are alike, and different from the third." This forced discrimination generates a new series of constructs, usually in the participant's own words. Finally, these constructs may be applied back to the original prompts, where the participant commonly uses a five-point Likert-style format to describe the extent to which each of the constructs relates to the original statement.

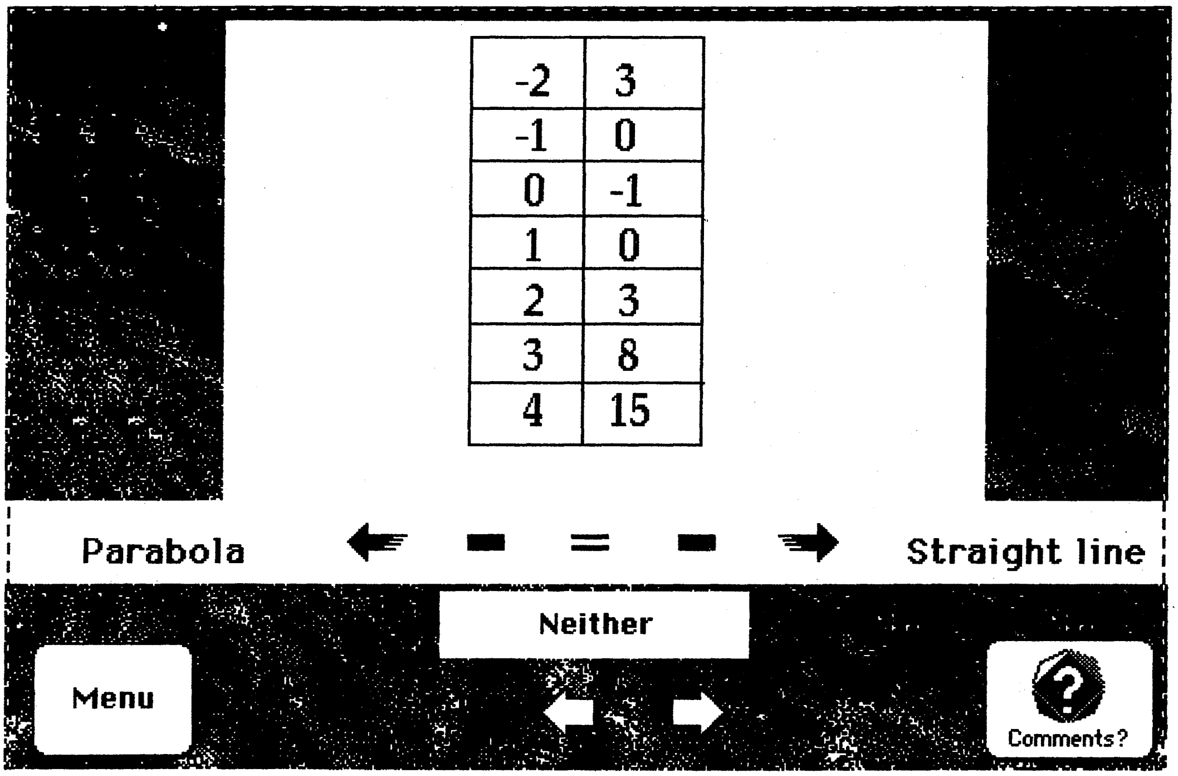
The study described in this paper extended this process to derive data on three distinct levels of increasing complexity. It also made use of the computer as the data-collection device, using *HyperCard* on the *Macintosh*. What are termed here "images of algebra" were elicited through presentation of a series of ten cards, displaying a range of common algebraic visual prompts: the *expression*, **4 - 3x**, the *equation*, **y = 2x - 1**, the *graph* of the parabola **y = (x-1)^2**, the *symbol*, **(x, y)**, the *symbol*, ***f*(x)**, the *graph* of the line **y = 2 - x**, the *table of values* for the relationship **y = x^2 - 1**, the *expression* **(x - 1)(x+1)**, the *equation*, **2x - 1 = x + 7** and the *graph* of **x = 2**. Participants responded in three ways:

(1) Respondents were first asked to verbally describe each card, and then to sort them into as many groupings as they could (this may be considered a *first order grouping*).

(2) Participants then engaged in a more detailed discriminatory exercise, in which the ten images were presented three at a time, and they were asked to "choose the odd one out" - to decide in what way one image was different to the other two. This *second order grouping* exercise forced participants to compare and contrast properties of the different images, and so potentially engage in a deeper analysis than the previous sorts.

(3) As a final, in-depth analytical mechanism, participants engaged in a *third order grouping,* a detailed Repertory Grid analysis of the categories which arose from the previous discriminatory activity. Categories identified from the second order grouping were taken in pairs, placed as the end points of a continuum, and then presented with each of the ten original images. For example, after distinguishing between, say, "parabola" and "straight line" in a second-order grouping, a respondent would then be asked to decide the extent to which each of the ten card images displayed these two properties by clicking at points between them (Figure 1). This process attempted to explicitly expose the network of relationships perceived by each individual in their thinking about algebra.

Figure 1: Sample of a *RepGrid* analysis card



Results

Images were chosen so as to offer the basis for sorts based both upon surface properties (for example, symbol/graph/numbers) and a range of possible other categories, such as functions/non-functions, different representations of the same symbolic form (cards 7 and 8) and even potential errors, such as equating the graph in card 2 with the visually similar symbolic forms of cards 7 and 8. Although the initial verbal/written responses of the participants was revealing of interesting aspects of their thinking, the focus in this paper lies with the three levels of the grouping process.

First Order Groupings.

A useful measure of the complexity of individual thinking about algebra was offered by the first order groupings of the image cards. This process had an immediacy which tended to be revealing of the *signal nature* of the various algebraic forms. Although at times the groupings were idiosyncratic, there were clear patterns of consistency which went beyond the "surface" characteristics of the cards.

S engaged twice in the algebra card sort activity and increased over the intervening period from four categories (*functions, solving equations, straight lines* and *parabolas*) to seven categories (*function, straight line, parabola, equation, coordinates, equation for axis* and *find values for variables*). His first sort was restricted in that he used each card only once, and so sorted them into exclusive categories. He displayed limited cross-representational facility, recognising the equation **y = 2x-1** as a linear graph and the expression **(x-1)(x+1)** as representing a parabola. He also treated the expression, **4 - 3x**, as an "implied equation", capable of solution if "**= 0**" is assumed as a suffix. Functions were included only as symbolic and numerical forms (the table of values implying for S an "input/output" image suggestive of function).

S's increased number of groupings appeared generally consistent but somewhat arbitrary (as in "Equation for axis"), and overall this sort demonstrated little improvement in his cognitive organisation beyond that which was made evident in the first. Although he showed good familiarity with the graphical representation, he appeared unable to interpret the table of values in a meaningful way.

T engaged in three first order sorts over a period of twelve months. Although the number of groupings increased in that period (from three to five), they remained based firmly upon superficial features of the images. In his first sort, **T** distinguished *equations, number plane* and *"things I don't understand."*

Interestingly, **T**'s second sort reduced the number of groups to two: *graphs* and *equations and algebra.* This second sort demonstrated improvement in both the appropriate use of technical terms ("graphs" instead of "number planes") and a clear distinction between what, for **T**, are the two fundamental divisions within algebra: symbols and graphs.

T's third sort (into 5 categories, *graph, equations, coordinates, table* and *expression*) displayed a finer detail and a better grasp of the language of algebra ("coordinates" and correct use of the term "expression") but not necessarily a deeper understanding of the distinctions between the various images. He now had a name for those "things I don't understand" from sort 1, and the expression **4 - 3x** had acquired the signal property of "something to be solved" which led to inclusion with the equations - although, once again, the other expression, **(x-1)(x+1)**, was omitted. **T** had studied expansion of binomials at school by this time, and indicated that he "knew what to do with this one", implying that he saw the expression **(x-1)(x+1)** as something to be expanded. It seems possible that the stronger "expansion" signal served to "swamp" the "equation" signal in this case. Note that he correctly placed the symbol ***f*(x)** as an expression, showing recognition of this form.

Finally, as a novice to algebra, **P**'s sorts would be expected to be based upon superficial cues, since the images possess for him no underlying meaning. His groupings reflect this: *graphical, numbers and letters, mix, loner* and *problems*.

Even at this early stage, **P** recognised that algebraic forms possess a signal nature. His last group, "problems," indicates objects which he saw as requiring some action, although he was unsure of what that action should be. His second sort was much more clearly defined than the first, with three clear and distinct groupings which again reflect the surface features of the algebraic forms: *graphs, equals* and *numerals.*

Second Order Groupings

Table 1

Second Order Groupings: *"Pick the odd one out"*

		S (Year 12)	T (Year 9)	P (Year 7)
1	4-3x y = 2x-1 Gr (y=2-x)	**1**	**1**	**3**
2	Gr (par) (x-1)(x+1) Table	**3**	**2**	**1**
3	(x, y) 2x-1=x+7 Table	**2**	**2**	**3**
4	f(x) 4 - 3x Gr (x=2)	**2**	**2**	**3**
5	(x-1)(x+1) 2x-1=x+7 4 - 3x	**1**	**1**	**3**
6	Gr (y=2-x) Gr (par) Gr (x=2)	**2**	**2**	**3**
7	y = 2x-1 (x, y) 2x-1 = x+7	**3**	**2**	**2**
8	y = 2x-1 f(x) 2x-1 = x+7	**3**	**2**	**2**
9	Gr (y=2-x) Gr (par) Table	**3**	**1**	**3**
10	f(x) (x, y) (x-1)(x+1)	**3**	**3**	**3**

The deliberate comparing and contrasting of algebraic images (picking the "odd one out") offers an added degree of depth to the analysis of participant responses, forcing them to go beyond the often-superficial viewing associated with a verbal description. In particular, respondents who had difficulty in supplying verbal descriptions were provided with a non-verbal means of conveying elements of their thinking about algebra. At the same time, these non-verbal responses were supplemented by comments regarding the choice made, which provided further insight into the reasons for these choices Table 8 summarises the participant responses for this task.

The sorting of the ten images into triads was, in most cases, deliberate rather than random. Each attempted to tease out a distinction which was seen by the researcher as significant, such as that between symbolic and graphical forms (triad 1), expressions and equations (triad 5) and others.

Third Order Groupings

The task which gave rise to the third order groupings was a very time-consuming one, taking up to thirty minutes to complete. For this reason, only the three participants were engaged in this activity - **S** (as the principal informant) and the two junior secondary students, **T** and **P** (whose limited algebraic experience meant that they had been restricted in their access to appropriate language and forms of expression by which their understanding might be examined). The previous tasks in these two cases had furnished limited information regarding their algebraic thinking - it was hoped that this detailed analysis might provide a useful non-verbal vehicle by which their cognitive frameworks could be better assessed.

This task involved three steps:

1. The verbal statements which had accompanied the second order grouping process were examined, and used to give rise to a number of descriptors which appeared to figure prominently in their thinking about the algebraic images. This process of extraction took place in collaboration with the informant, increasing validity for the descriptors (Table 2).

2. The descriptors were entered into the *HyperCard* RepGrid stack and participants again viewed each card individually. This time, however, instead of requiring a verbal descriptor, the descriptors were displayed in pairs, as the ends of a continuum (see Figure 1).

3. Participants chose an appropriate response which situated the given image in relation to the two descriptors.

Table 2:
Descriptors from the second order grouping task

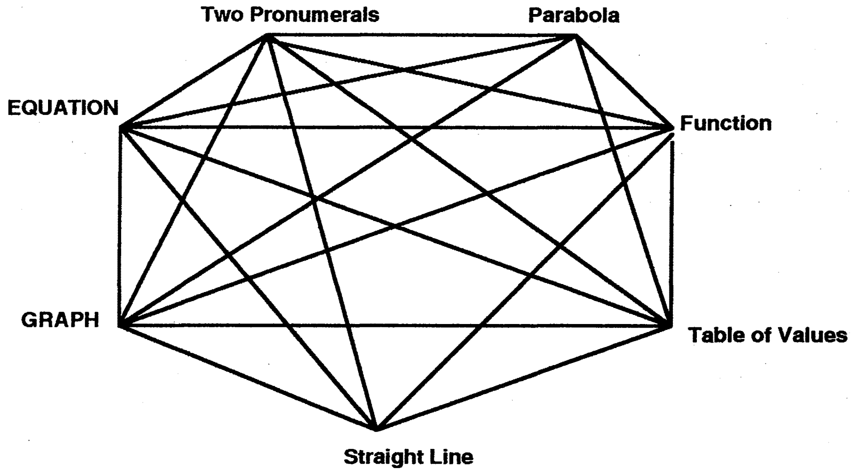
S	T	P
• two pronumerals	• graph	• numerals
• parabola	• coordinates	• picture
• function	• table	• not sure
• straight line	• x and y	• graph
• graph	• equations	• equals
• table of values	• expansion	• coordinations
• equation	• parabola	• symbol
	• straight line	

The networks of relationships derived from this task proved highly informative regarding the algebraic thinking of the various individuals involved. While the previous sorting tasks had allowed the identification of the various categories by which algebraic objects were conceptualised, this final task allowed these categories to be located within a dimensional space.

Summary diagrams of these concept networks were developed. Heavy lines indicate a strong link between the two descriptors (defined as four or more occasions when these were deliberately related by the respondent). Lighter lines indicate weaker links (less than four occurrences).

S's network of concepts displays well-developed links between all major categories, suggesting good algebraic understanding. His graphical thinking appears to be better developed than that associated with the table of values, with EQUATIONS and TWO PRONUMERALS being most extensively related. **S**'s concept links appear to be of the "all or none" kind, suggesting that he distinguishes less than clearly between them (as in his use of the terms "function" and "equation").

Figure 2: Concept Network for **S**



The differences between the concept network for **S** and those for **T** and **P** are immediately clear. While the younger students might have identified as many descriptors, these are poorly developed and associated constructs. Their relationships with other concepts is tenuous at best, illustrative of multistructural understanding at best.

Figure 3: Concept Network for **T**

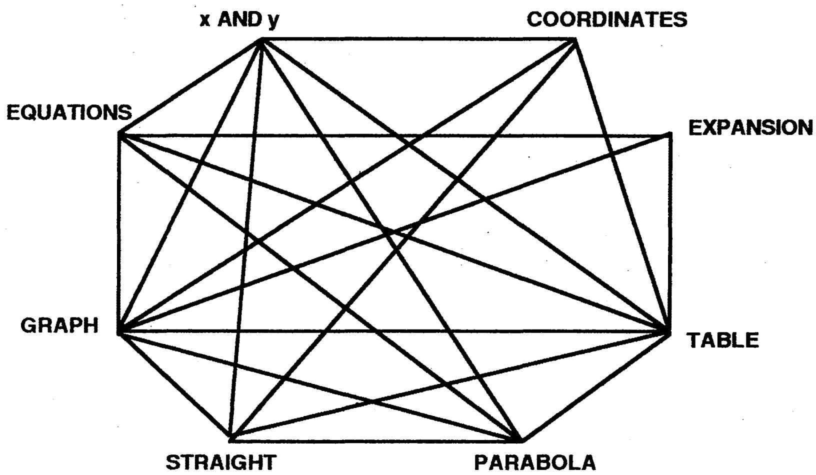
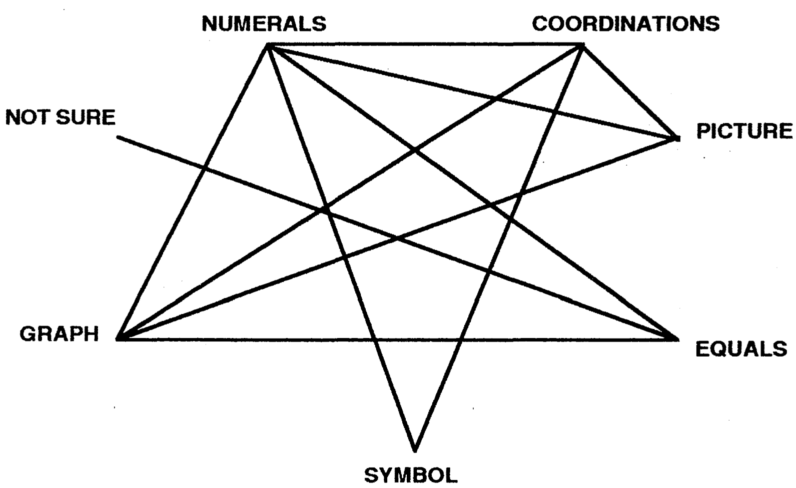


Figure 4: Concept Network for **P**



It is hardly surprising to find **P**'s concept network to be even more limited than that of **T**, sure only that algebra involves pictures, graphs, numerals and symbols. Like **T**, repeated descriptors may be recognised: "x and y" and "coordinates" for **T**, "pictures" and "graphs" for **P**. **P**'s few strong links are those between NUMERALS and SYMBOLS, and PICTURES and GRAPHS (demonstrating the symbol nature of algebraic forms). Clearly, for **T**, graphs are more meaningful objects than for **P**, extending even to the recognition of the symbolic connections between equations and expressions and their graphical forms.

Conclusion

The network diagrams provided immediate visual clues as to the cognitive organisation of the participants. They illustrated both the nature and the relative strength of the relationships between the various constructs which made up each individual's cognitive network within the domain of algebra. Across the participants involved in the study, it was now possible to recognise important and detailed features of their algebraic thinking as a result of the intricate examination which has been described. The cognitive profiles which were developed offered valuable insights into the algebraic thinking of these individuals and guidance in the further study of the role of computer tools in the algebra learning process. The matrices which give rise to these networks of relationships are amenable to statistical analysis if desired and potentially the method offers a relatively simple but detailed means of studying the ways in which individuals think about significant mathematical concepts which, all too often, are far more image-laden than they are verbal for learners.

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